LECTURE

7

RIEMANN INTEGRALS. IMPROPER INTEGRALS

Riemann integrals

In what follows we assume that a, b ∈ R, a<b.

Definition 7.1 A partition of [a, b] is a finite ordered set P = (x0,x1,...,xn) of numbers s.t.

a = x0 < x1 <...<xn−1 < xn = b.

By a subinterval of P we mean any interval [xi−1,xi] with i ∈ {1,...,n}. The norm of P is the length of the largest subinterval of P, i.e.,

P .= .max{xi − xi−1 | i = 1,n}.

If ξ .= .(ξ1,...,ξn) is an ordered set of real numbers such that

ξi ∈ [xi−1,xi], ∀i ∈ {1,...,n},

then (P, ξ) is called a tagged partition of [a, b].

Definition 7.2 Let f : [a, b] → R be a function. By the the Riemann sum of f with respect to a tagged partition (P, ξ) of [a, b], we mean

σ(f,P,ξ) =

∑ni=1

f(ξi)(xi − xi−1).

f(ξi)(xi − xi−1).

Definition 7.3 A function f : [a, b] → R is said to be Riemann integrable on [a, b] if there exists I ∈ R satisfying the following condition:

∀ε > 0, ∃δ > 0 s.t. |σ(f,P,ξ) − I| < ε,∀(P, ξ) tagged partition with P < δ.

The family of all Riemann integrable functions on [a, b] is denoted by R[a, b].

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Remark 7.4 (i) If f ∈ R[a, b], then I ∈ R satisfying the required condition in Definition 7.3 is uniquely determined and called the Riemann integral (or definite integral) of f on [a, b]. We denote

∫ ba f(x)dx =

∫ ba f ..= I.

(ii) If f : [a, b] → R+ and f ∈ R[a, b], then A =

∫ ba f is the area of the set

(hypof) ∩ (R × R+) = {(x, y) ∈ R2 | x ∈ [a, b], 0 ≤ y ≤ f(x)}

located under the graph of f above the axis 0x. (iii) If f : [a, b] → R is continuous, then f ∈ R[a, b]. (iv) If f : [a, b] → R is monotone, then f ∈ R[a, b]. (v) If f ∈ R[a, b], then f is bounded.

Theorem 7.5 For any f,g ∈ R[a, b] and α ∈ R we have:

(i) f + g ∈ R[a, b] and

∫ ba (f + g) =

∫ ba f +

∫ ba g.

(ii) (αf) ∈ R[a, b] and

∫ ba (αf) = α∫ ba f. (iii) (f · g) ∈ R[a, b].

(iv) |f| ∈ R[a, b].

(v) If f ≤ g, then

∫ ba f ≤

∫ ba g.

Theorem 7.6 Let f : [a, b] → R and c ∈ (a, b). Then

f ∈ R[a, b] ⇐⇒ f|[a,c] ∈ R[a, c] and f|[c,b] ∈ R[c, b].

In this case,

∫ ba f =

∫ ca f +

∫ b

c f.

Theorem 7.7 (First Fundamental Theorem of Calculus) Let f ∈ R[a, b]. Define the function F : [a, b] → R for all t ∈ [a, b] by

F(t) .=

.∫ ta f. Then F is continuous. Moreover, if f is continuous at c ∈ [a, b], then F is differentiable at c and F (c) = f(c).

Theorem 7.8 (Second Fundamental Theorem of Calculus) If f ∈ R[a, b] and F : [a, b] → R is an antiderivative of f (that is, F (x) = f(x), ∀x ∈ [a, b]), then the Leibniz-Newton Formula holds: ∫ ba f = F(b) − F(a).

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Improper integrals

Remark 7.9 Consider the function f : [0,1) → R,

f(x) .= .√1 1 − x2. Note that x = 1 is a vertical asymptote of f and hence the question of how one could define the area under the graph of f arises. To this end, let t ∈ [0,1) and f|[0,t]. Then

At =

∫ t0

√1 1 − x2dx = arcsint

is the area under the graph of f|[0,t]. One can now define the area under the graph of f as

A = lim t→1 t<1 At = π2.

In a similar way one treats the problem for the function f : [1,+∞) → R,

f(x) ..= x12.

For t ∈ [1,+∞), At =

∫ t1

x12dx = 1 − 1t and so A = t→∞lim At = 1.

Definition 7.10 Let f : I → R be a function defined on an interval I ⊆ R. We say that f is locally Riemann integrable on I if for all a, b ∈ I with a<b the function f|[a,b] is Riemann integrable on [a, b].

Remark 7.11 (i) If f ∈ R[a, b], then f is locally Riemann integrable on [a, b]. (ii) If f : R → R is continuous, then f is locally Riemann integrable on R.

Definition 7.12 Let a, b ∈ R with a < b and let f : [a, b) → R be a function, which is locally Riemann integrable on [a, b). If the following limit exists in R, then it is called the improper integral of f on [a, b):

∫ ba f(x)dx .=

.∫ b−0

a f(x)dx .= .lim t→b t<b

∫ ta f(x)dx.

We say that the improper integral

∫ b−0

a f(x)dx is convergent if it is finite; in this case, f is said to be improperly integrable on [a, b). Otherwise, we say that the improper integral

∫ b−0

a f(x)dx is divergent.

Definition 7.13 Let a ∈ R and let f : [a,+∞) → R be a function, which is locally Riemann integrable on [a,+∞). If the following limit exists in R, then it is called the improper integral of f on [a,+∞):

∫ +∞

a f(x)dx .= .t→+∞lim ∫ ta f(x)dx.

We say that the improper integral

∫ +∞

a f(x)dx is convergent if it is finite; in this case, f is said to be improperly integrable on [a,+∞). Otherwise, we say that the improper integral

∫ +∞

a f(x)dx is divergent.

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Definition 7.14 Let a, b ∈ R with a < b and let f : (a, b] → R be a function, which is locally Riemann integrable on (a, b]. If the following limit exists in R, then it is called the improper integral of f on (a, b]:

∫ ba f(x)dx ..=

∫ ba+0 f(x)dx ..= lim t→a t>a

∫ b

t f(x)dx.

We say that the improper integral

∫ ba+0 f(x)dx is convergent if it is finite; in this case, f is said

to be improperly integrable on (a, b]. Otherwise, we say that the improper integral

∫ ba+0 f(x)dx is divergent.

Definition 7.15 Let b ∈ R and let f : (−∞,b] → R be a function, which is locally Riemann integrable on (−∞,b]. If the following limit exists in R, then it is called the improper integral of f on (−∞,b]:

∫ b−∞ f(x)dx .= .t→−∞lim ∫ b

t f(x)dx.

We say that the improper integral

∫ b−∞ f(x)dx is convergent if it is finite; in this case, f is said

to be improperly integrable on (−∞,b]. Otherwise, we say that the improper integral

∫ b−∞ f(x)dx is divergent.

Definition Riemann and f 7.16 Let a, b ∈ R with a < b and let f : (a, b) → R ∫ bon c (a, integrable on (a, b). If are convergent (i.e., f|(a,c] b) is defined as:

there exists c ∈ (a, b) such be a that both improper function, integrals which ∫ is a cf(x)dx locally

and f|[c,b) are improperly integrable), then the improper integral of

∫ ba f(x)dx =

∫ ca f(x)dx +

∫ b

c f(x)dx

Remark 7.17 There exists a close connection between the improper integrals on intervals of type [a,+∞) and the series of real numbers.

Theorem 7.18 (Cauchy’s Integral Test for Convergence of Series) Let f : [m,+∞) → [0,+∞)

be a decreasing function, where m ∈ N. Then the improper integral

∫ +∞

m f(x)dx is convergent if and only if the series ∑n≥mf(n) is convergent.

Example 7.19 (The generalized harmonic series) For ∑n≥1

1nα with α > 0, let us define the function f : [1,+∞) → [0,+∞), fact that the generalized harmonic f(x) series = converges x1α. According to the Integral Test we recover the known

for α > 1 and diverges for 0 < α ≤ 1.

Theorem 7.20 (Comparison Test for Improper Integrals) Let a ∈ R and b ∈ R∪{+∞} with a<b and let f,g : [a, b) → R be locally Riemann integrable functions, such that

∃c ∈ [a, b) s.t. ∀x ∈ [c, b), 0 ≤ f(x) ≤ g(x). (7.1)

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Then the following assertions hold true:

1◦ If the improper integral

∫ ba g(x)dx is convergent,

then the improper integral

∫ ba f(x)dx is convergent.

2◦ If the improper integral

∫ ba f(x)dx is divergent, then the improper integral

∫ ba g(x)dx g is divergent.

Remark 7.21 If f and g in the above theorem are nonnegative locally Riemann integrable functions on [a, b) satisfying the following condition

∃α,β > 0,∃c ∈ [a, b) s.t. ∀x ∈ [c, b), αg(x) ≤ f(x) ≤ βg(x),

then the improper integrals

∫ ba f(x)dx and

∫ ba g(x)dx have the same nature.

Corollary 7.22 Let a, b ∈ R with a < b, f : [a, b) → [0,+∞) be a locally Riemann integrable function on [a, b) and p ∈ R such that the following limit exists in R:

L .= .x→b lim x<b(b − x)pf(x).

Then the following assertions hold true:

1◦ If p < 1 and L < +∞, then the improper integral

∫ b−0

a f(x)dx is convergent.

2◦ If p ≥ 1 and L > 0, then the improper integral

∫ b−0

a f(x)dx is divergent.

Proof. 1◦ By definition of L, there exists c ∈ [a, b) such that

∀x ∈ [c, b), (b − x)pf(x) < L + 1.

Thus,

∀x ∈ [c, b), 0 ≤ f(x) < (b L − + x)1

p.

Take g : [a, b) → R, g(x) = (b L − + x)1

p. Since p < 1, the improper integral

∫ b−0

a g(x)dx is convergent.

By Theorem 7.20 (1◦) it follows that the improper integral

∫ b−0

a f(x)dx is convergent. 2◦ Let r ∈ (0,L). By definition of L, there exists c ∈ [a, b) such that

∀x ∈ [c, b),r< (b − x)pf(x).

Thus, we have

∀x ∈ [c, b), 0 < r

(b − x)p < f(x).

Take h : [a, b) → R, h(x) = (b − r

x)p. Since p ≥ 1, the improper integral

∫ b−0

a h(x)dx is divergent.

Applying Theorem 7.20 (2◦), we conclude that the improper integral

∫ b−0

a f(x)dx is divergent. D

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Corollary 7.23 Let a, b ∈ R with a < b, f : (a, b] → [0,+∞) be a locally Riemann integrable function on [a, b) and p ∈ R such that the following limit exists in R:

L .= .x→a lim x>a(x − a)pf(x).

Then the following assertions hold true:

1◦ If p < 1 and L < +∞, then the improper integral

∫ ba+0 f(x)dx is convergent.

2◦ If p ≥ 1 and L > 0, then the improper integral

∫ ba+0 f(x)dx is divergent.

Corollary 7.24 Let a ∈ R, f : [a,+∞) → [0,+∞) be a locally Riemann integrable function on [a,+∞) and p ∈ R such that the following limit exists in R:

L .= .x→∞lim xpf(x).

Then the following assertions hold true:

1◦ If p > 1 and L < +∞, then the improper integral

∫ +∞

a f(x)dx is convergent.

2◦ If p ≤ 1 and L > 0, then the improper integral

∫ +∞

a f(x)dx is divergent.

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